Optimum Time Step and Memory for GPS-based Unbiased FIR Estimates of the Local Clock TIE Model

Yuriy S. Shmaliy, Jorge Munoz-Diaz, Luis Arceo-Miquel, and Oscar Ibarra-Manzano
FIMEE, Guanajuato University, Salamanca, Gto, 36730, Mexico
Email: shmaliy@salamanca.ugto.mx

***Abstract*— In this paper, we address a theoretical analysis of errors of the l-degree unbiased finite impulse response (FIR) filter applied to the K-degree time interval error (TIE) model of a local clock. Emphasizing the fact that the TIE model originates from the slowly changing Brownian phase and thus cannot obligatorily be distinct on a horizon of N points, we investigate estimates for $l \neq K$ and derive optimal horizons $N_{\mathrm{opt}}(\tau_{\mathrm{opt}})$ as functions of the optimal time step τ_{opt}. Practical justification is provided based upon GPS-based sawtooth measurements of the TIE of a crystal clock imbedded to the Stanford Frequency Counter SR620 and using the SynPaQ III GPS Sensor and Symmetricom cesium standard of frequency as a reference source.**

I. Introduction

The time interval error (TIE) of a local clock originates from the phase behavior of a clock oscillator. When white frequency noise dominates, the TIE behaves as the Brownian phase and can be simulated as the Wiener process. Other kinds of noise complicate dynamic picture so that the TIE is measured to be a nonstationary random function that is extremely narrowband as generated by precision clocks having oscillators with extra-high quality factors. Owing to stochastic nonstationarity, the TIE model cannot be identified precisely on a horizon of N points to design optimum estimators, although it is known to be near uniform for cesium clocks, linear for rubidium ones, and linear or quadratic for crystal clocks.

Measurements of the TIE are often provided using commercially available Global Positioning System (GPS) timing receivers. The TIE $x(n)$ is measured with the time step $\tau = 1$s as a difference between the GPS time and the local clock time in the presence of typically the sawtooth noise $v(n)$ owing to the principle of the one pulse per second (1PPS) signal formation utilized to the receiver. The standard deviation of $v(n)$ using commercially available receivers is about 30 ns, can reach 10-20 ns [1] and may be improved by removal of systematic errors to no less than 3-5 ns [1], [2].

In modern receivers, such as the Motorola M12+ [4] and SynPaQ III GPS Sensor, uniformly distributed the sawtooth noise $v(n)$ ranges within the bounds $\pm\Delta[\mathrm{ns}] = 10^3/2f_{\mathrm{LTC}}[\mathrm{MHz}]$, where f_{LTC} is the frequency of a Local Time Clock (LTC) of the receiver [3]. It can be shown that the sawtooth structure of $v(n)$ represents the modulo 2Δ near Brownian TIE associated with the phase of the LTC. In the SynPaQ III GPS Sensor and Motorola M12+ GPS timing receiver, the frequency is chosen to be $f_{\mathrm{LTC}} = 10$ MHz and the bound is thus $\Delta = 50$ ns. In a modified M12+, they let $f_{\mathrm{LTC}} = 40$ MHz [5] and hence $\Delta = 12.5$ ns.

To estimate the TIE model with high accuracy, the unbiased FIR filter was designed in [6], [7]. For the linear TIE, this filter was first derived in [6] as an optimally unbiased moving average filter. The latter was then examined in [8] for applications in GALILEO and it was experimentally shown (Fig. 8 in [8]) that the mean square error (MSE) and Allan deviation in its estimates are both minimum among other filters examined. Under model uncertainty, an optimal averaging horizon N_{opt} of the unbiased FIR filter strongly depends on the time step τ obtained by thinning the 1PPS pulses [9].

In this paper, we provide a theoretical analysis of $N_{\mathrm{opt}}(\tau_{\mathrm{opt}})$ and compare the results to the values experimentally found in [9]. We show that prediction fits measurements in both cases examined: when simple averaging is applied to a linear TIE model and a linear unbiased FIR filter estimates a quadratic TIE model.

II. Unbiased FIR Estimates of the TIE

Below, we begin with a succinct discussion of the TIE polynomial model of a local clock and proceed with its unbiased FIR estimates.

A. TIE Model of a Clock

The TIE polynomial model of a clock projects ahead on a horizon of N points from the start point $n = 0$ with the K-degree Taylor polynomial

$$x_1(n) = \sum_{p=0}^{K} x_{p+1}\frac{\tau^p n^p}{p!} + w_1(n,\tau)$$

$$= x_1 + x_2\tau n + \frac{x_3}{2}\tau^2 n^2 + \frac{x_4}{6}\tau^3 n^3 \ldots + w_1(n,\tau)\,, \quad (1)$$

where $x_k \equiv x_k(0)$, $k \in [1, K+1]$, are initial states of the clock and $w_1(n,\tau)$ is a clock noise with known properties. By extending the time derivatives of the TIE model to the Taylor series, the signal and observation equations become, respectively,

$$\lambda(n) = \mathbf{A}(n)\lambda(0) + \mathbf{w}(n,\tau)\,, \quad (2)$$

1-4244-0647-1/07/$20.00

$$\xi(n) = \mathbf{C}\lambda(n) + \mathbf{v}(n)\,, \tag{3}$$

where

$$\lambda(n) = [\, x_1(n)\, x_2(n) \, \dots \, x_{K+1}(n) \,]^T \tag{4}$$

is a $(K+1)\times 1$ vector of the clock states and a $(K+1) \times (K+1)$ time-varying system matrix is

$$\mathbf{A}(n) = \begin{bmatrix} 1 & \tau n & \tau^2 n^2/2 & \dots & (\tau n)^K/K! \\ 0 & 1 & \tau n & \dots & (\tau n)^{K-1}/(K-1)! \\ 0 & 0 & 1 & & (\tau n)^{K-2}/(K-2)! \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{5}$$

For equal numbers of states and measurements, the observation vector is formed as

$$\xi(n) = [\, \xi_1(n)\, \xi_2(n) \, \dots \, \xi_{K+1}(n) \,]^T \tag{6}$$

and the $(K+1)\times(K+1)$ measurement matrix $\mathbf{C}$ is typically identity. The $(K+1)\times 1$ clock noise vector is described by

$$\mathbf{w}(n,\tau) = [\, w_1(n,\tau)\, w_2(n,\tau) \, \dots \, w_{K+1}(n,\tau) \,]^T \tag{7}$$

with the components caused by clock noises affecting the states. We notice that each of the components in (7) typically depends on the time step τ. Finally, the measurement noise vector

$$\mathbf{v}(n) = [\, v_1(n)\, v_2(n) \, \dots \, v_{K+1}(n) \,]^T \tag{8}$$

contains correlated or uncorrelated components that are not always Gaussian.

B. Unbiased FIR Estimates of the TIE Model

To provide real-time estimation of $\lambda(n)$ and obtain the $(K+1)\times 1$ vector

$$\hat{\lambda}(n) = [\, \hat{x}_1(n)\, \hat{x}_2(n) \, \dots \, \hat{x}_{K+1}(n) \,]^T \tag{9}$$

of the estimates $\hat{x}_k(n)$, $k \in [1, K+1]$, of the clock states $x_k(n)$, the unbiased FIR filtering algorithm [7] can be used. By this algorithm, the unknown kth component, $k \geq 2$, of the observation vector (6) is formed as the backward discrete time derivative of the estimate $\hat{x}_{k-1}(n)$,

$$\xi_k(n) = \frac{\hat{x}_{k-1}(n) - \hat{x}_{k-1}(n-1)}{\tau}\,. \tag{10}$$

Utilizing N points of the nearest past, an unbiased FIR estimate $\hat{x}_k(n)$ of $x_k(n)$ is obtained by the discrete-time convolution operator $\mathcal{C}_k$ applied to (10) as

$$\hat{x}_k(n) = \mathcal{C}_k \xi_k(n) = \sum_{i=0}^{N-1} h_l(i,N)\xi_k(n-i) \tag{11a}$$

$$= \mathbf{W}_l^T(N)\mathbf{\Xi}_k(n)\,, \tag{11b}$$

where $l \geq 0$ is the FIR filter degree that, if the TIE model degree K is distinct, must be set to be $K-k+1$ in order to produce an unbiased estimate. Here the generic FIR component $h_l(i,N)$ has inherent properties

$$h_l(i,N) = \begin{cases} h_l(i,N), & 0 \leq i \leq N-1 \\ 0, & \text{otherwise} \end{cases}, \tag{12}$$

$$\sum_{i=0}^{N-1} h_l(i,N) = 1\,, \tag{13}$$

the l-degree FIR $N \times 1$ matrix is given with

$$\mathbf{W}_l(n) = [\, h_l(0)\, h_l(1) \, \dots \, h_l(N-1) \,]^T\,, \tag{14}$$

and the $N \times 1$ matrix of the kth state measurements is

$$\mathbf{\Xi}_k(n) = [\, \xi_k(n)\, \xi_k(n-1) \, \dots \, \xi_k(n-N+1) \,]^T. \tag{15}$$

The vector (9) is thus defined as

$$\hat{\lambda}(n) = \mathcal{C}\xi(n)\,, \tag{16}$$

where the $(K+1)\times(K+1)$ convolution operator matrix is

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_1 & 0 & \dots & 0 \\ 0 & \mathcal{C}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{C}_{K+1} \end{bmatrix}, \tag{17}$$

having the component $\mathcal{C}_k$, $k \in [1, K+1]$, specified by (11a).

C. Unique Unbiased FIRs for Real Time Estimation

The unique FIR for real time unbiased estimation (11a) was derived in [7] in the form of

$$h_l(i) = \sum_{j=0}^{l} a_{jl} i^j, \tag{18}$$

where a generic coefficient a_{jl} is determined by

$$a_{jl} = (-1)^j \frac{M_{(j+1)1}}{|\mathbf{D}|} \tag{19}$$

via the determinant $|\mathbf{D}|$ and minor $M_{(j+1)1}$ of the $(l+1)\times(l+1)$ quadratic matrix $\mathbf{D}$,

$$\mathbf{D} = \begin{bmatrix} d_0 & d_1 & d_2 & \dots & d_l \\ d_1 & d_2 & d_3 & \dots & d_{l+1} \\ d_2 & d_3 & d_4 & \dots & d_{l+2} \\ \vdots & \vdots & \vdots & \ddots & \\ d_l & d_{l+1} & d_{l+2} & \dots & d_{2l} \end{bmatrix}, \tag{20}$$

which component d_m, $m \in [0, 2l]$, is calculated by the Bernoulli polynomials $B_n(x)$ as

$$d_m = \sum_{i=0}^{N_l-1} i^m = \frac{1}{m+1}\left[B_{m+1}(N) - B_{m+1}\right], \tag{21}$$

where $B_n = B_n(0)$ is the Bernoulli coefficient.

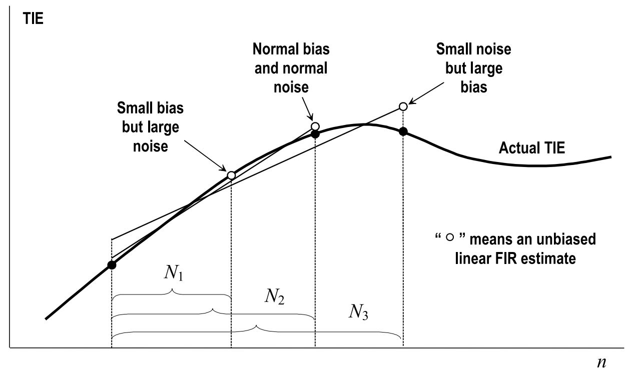

Fig. 1. Errors in the unbiased linear FIR estimates with the nonlinear TIE model.

In time scales, the degree K is identified for the filter memory on a horizon of N points by the clock precision. Typically, $K = 0$ fits cesium clocks, $K = 1$ is appropriate for rubidium clocks, and $K \in [1, 2]$ for crystal clocks. However, $K = 3$ may be required for low precision crystal clocks. For all these cases, the unique FIRs were found in [7] and postponed to Table I.

By (11b) and $h_l(i, N)$ (Table I), the kth clock state is estimated with highest accuracy in the sense of the minimum produced bias if the filter degree is set such that $l = K - k + 1$.

III. Optimum N and τ for Unbiased FIR Estimates of the TIE

It is now a proper place to discuss errors in the unbiased FIR estimates. Seemingly obvious is that if the TIE behaves absolutely linearly ($K = 1$) or quadratically ($K = 2$), or cubically ($K = 3$), etc. over all time, then the filter with $l = 1$, $l = 2$, and $l = 3$ (Table I), respectively, will produce zero bias and, by $N \to \infty$, zero noise. Thus the estimate would be optimum in the sense of the minimum MSE. In a reality, the TIE is stochastic and nonstationary. Therefore, the filter is able to fit the model only on a finite horizon of N points.

Fig. 1 gives an idea about the estimate error if we apply a linear filter (Table I, $l = 1$) to a slowly changing TIE function. On a short horizon N_1, this filter fairly fits the behavior with a small bias in the estimate. The estimate noise, however, is large here, because N_1 is small. With some N_2, both the bias and noise variance are small. That is what to say an optimal solution in the sense of the minimum MSE. In the last case of N_3, the filter no longer fits the TIE owing to large bias and in spite of small noise in the estimate.

In other words, under the model uncertainty an unbiased FIR filter of any degree l must be optimized for the horizon N and time step τ, because noise is τ-dependent [9], [10].

A. MSE of the Unbiased FIR Estimate

We now will think that any of the clock states is described with the K-degree polynomial (1). If we assign $z_K(n)$ to represent $x_k(n)$, $k \in [1, K+1]$, then, by (11b) and (3), an unbiased FIR estimate of z_K can be found as

$$\hat{z}_K(n) = \mathbf{W}_l^T \mathbf{z}_K(n) + \mathbf{W}_l^T \mathbf{u}_K(n)\,, \tag{22}$$

where $\mathbf{W}_l(n)$ is specified by (14) and Table I. The $N \times 1$ matrices of the clock state $z_K(n)$ and measurement noise $u_K(N)$ representing $v_k(n)$, $k \in [1, K+1]$, are given by, respectively,

$$\mathbf{z}_K(n) = [\, z_K(n)\, z_K(n-1)\, \ldots\, z_K(n-N+1)\,]^T, \tag{23}$$

$$\mathbf{u}_K(n) = [\, u_K(n)\, u_K(n-1)\, \ldots\, u_K(n-N+1)\,]^T, \tag{24}$$

where $u_K(n)$ is zero-mean with a variance σ_{uK}^2.

The estimate error can be evaluated as

$$\varepsilon_K(n) = \hat{z}_K(n) - z_K(n) \tag{25}$$

and the relevant MSE by averaging the square of (25) over all available estimates. We thus have

$$\langle \varepsilon_K^2 \rangle = \langle [\hat{z}_K(n) - z_K(n)]^2 \rangle \tag{26a}$$

$$= \langle [\mathbf{W}_l^T \mathbf{z}_K(n) + \mathbf{W}_l^T \mathbf{u}_K(n) - z_K(n)]^2 \rangle \tag{26b}$$

$$= [\mathbf{W}_l^T \mathbf{z}_K(n) - z_K(n)]^2 + \langle [\mathbf{W}_l^T \mathbf{u}_K(n)]^2 \rangle\,. \tag{26c}$$

Because $\mathbf{W}_l^T \mathbf{u}_K(n)$ represents an amount of noise $u_K(n)$ at the output of the filter of degree l, we can introduce a new noise $\bar{u}_{Kl}(n) = \mathbf{W}_l^T \mathbf{u}_K(n)$ that is zero-mean, $\langle \bar{u}_{Kl}(n) \rangle = 0$, with the variance $\sigma_{Kl}^2 = \langle [\mathbf{W}_l^T \mathbf{u}_K(n)]^2 \rangle$. By large N, noise well normalizes [7] that allows us to represent σ_{Kl}^2 via the noise-power gain $g_l(N)$ as

$$\sigma_{Kl}^2 = g_l(N) \sigma_{uK}^2\,. \tag{27}$$

For $l \in [0, 3]$, the noise-power gain is given in Table I [7].

On the other hand, the relation in brackets of the first term in (26c) represents the estimate bias

$$\mathrm{bias}_{Kl}(n) = \mathbf{W}_l^T \mathbf{z}_K(n) - z_K(n)\,. \tag{28}$$

If we think that the bias is constant on a horizon N, then the MSE (26c) can be rewritten as

$$\langle \varepsilon_{Kl}^2 \rangle = \mathrm{bias}_{Kl}^2 + \sigma_{Kl}^2 \tag{29}$$

and, by accounting for dependencies on N and τ, we have

$$\langle \varepsilon_{Kl}^2(\tau, N) \rangle = \mathrm{bias}_{Kl}^2(\tau, N) + \sigma_{Kl}^2(\tau, N)\,. \tag{30}$$

B. Optimum N and τ

To define optimum values of τ and N in the sense of the minimum MSE, we first minimize (30) for N by equating to zero the derivative of $\langle \varepsilon_{Kl}^2(\tau, N) \rangle$ taken with respect to N,

$$\frac{\partial}{\partial N} \langle \varepsilon_{Kl}^2(\tau, N) \rangle = \frac{\partial}{\partial N} \mathrm{bias}_{Kl}^2(\tau, N) + \frac{\partial}{\partial N} \sigma_{Kl}^2(\tau, N) = 0\,. \tag{31}$$

A solution of (31) gives a function $N_{\mathrm{opt}}(\tau)$.

Reasoning similarly for τ, we write

TABLE I

UNIQUE IMPULSE RESPONSE $h_l(i,N)$ AND NOISE-POWER GAIN $g_l(N)$ OF AN UNBIASED FIR FILTER FOR $l \in [0,3]$.

l	FIR	$h_l(i,N)$	$g_l(N)$
0	Uniform	$\frac{1}{N}$	$\frac{1}{N}$
1	Linear	$\frac{2(2N-1)-6i}{N(N+1)}$	$\frac{2(2N-1)}{N(N+1)}$
2	Quadratic	$\frac{3(3N^2-3N+2)-18(2N-1)i+30i^2}{N(N+1)(N+2)}$	$\frac{3(3N^2-3N+2)}{N(N+1)(N+2)}$
3	Cubic	$\frac{8(2N^3-3N^2+7N-3)-20(6N^2-6N+5)i+120(2N-1)i^2-140i^3}{N(N+1)(N+2)(N+3)}$	$\frac{8(2N^3-3N^2+7N-3)}{N(N+1)(N+2)(N+3)}$

$$\frac{\partial}{\partial\tau}\langle\varepsilon^2_{Kl}(\tau,N)\rangle = \frac{\partial}{\partial\tau}\text{bias}^2_{Kl}(\tau,N) + \frac{\partial}{\partial\tau}\sigma^2_{Kl}(\tau,N) = 0 \tag{32}$$

and solve (32) in order to find τ_{opt}. By this value, we finally have $N_{\text{opt}}(\tau_{\text{opt}})$.

Relations (31) and (32) hold true for any FIR filter and TIE model. In the limiting, albeit isolated, case when the unbiased filter fits the model over all N, bias inherently become zero and τ_{opt} and N_{opt} are derived from

$$\left.\frac{\partial}{\partial N}\sigma^2_{Kl}(\tau,N)\right|_{N=N_{\text{opt}}(\tau)} = 0\,, \tag{33}$$

$$\left.\frac{\partial}{\partial\tau}\sigma^2_{Kl}(\tau,N)\right|_{\tau=\tau_{\text{opt}}} = 0\,. \tag{34}$$

The latter may be illustrated by Fig. 1 is we suppose that the TIE model is linear over all n, unlike a nonlinear one as shown. However, the linear behavior contradicts the Brownian origin of the TIE and basic equations to find N_{opt} and τ_{opt} still remain (31) and (32), respectively. An additional observation reveals that, in order to find N_{opt} and τ_{opt}, it is enough observing only parts of the TIE behavior at which the filter does not fit the TIE model critically, as shown in Fig. 1.

IV. ESTIMATING N_{opt} AND τ_{opt}

Below, we find and examine N_{opt} and τ_{opt} for two most common particular cases. We first apply simple averaging ($l = 0$) to the linear model ($K = 1$) that can be met, for example, in atomic clocks with a small amount of frequency offset. Next, the unbiased linear FIR filter ($l = 1$) is applied to the presumingly linear TIE model of a crystal clock that tends to be quadratic ($K = 2$), by large N. A comparison is provided of the predicted N_{opt} and τ_{opt} to the relevant values found experimentally.

We relate our investigations to the zero-mean sawtooth noise induced by the GPS timing receiver. This noise is uniformly distributed [3] within the bounds $\pm\Delta$ having the variance

$$\sigma^2_{uK} = \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} z^2 dz = \frac{\Delta^2}{3}\,. \tag{35}$$

By $g_l(N)$ (Table I) and (27), the estimate variance σ^2_{Kl} is defined and postponed to Table II.

TABLE II

VARIANCE OF THE UNBIASED FIR ESTIMATE FOR SAWTOOTH NOISE.

l	FIR	σ^2_{Kl}
0	Uniform	$\frac{\Delta^2}{3N}$
1	Linear	$\left.\frac{\Delta^2 2(2N-1)}{3N(N+1)}\right\|_{N\gg1} \cong \frac{4\Delta^2}{3N}$
2	Quadratic	$\left.\frac{\Delta^2 3(3N^2-3N+2)}{3N(N+1)(N+2)}\right\|_{N\gg1} \cong \frac{3\Delta^2}{N}$
3	Cubic	$\left.\frac{\Delta^2 8(2N^3-3N^2+7N-3)}{3N(N+1)(N+2)(N+3)}\right\|_{N\gg1} \cong \frac{16\Delta^2}{3N}$

To evaluate the bias in the estimate for given K and l, it need substituting $z_K(n)$ described by (1), (14), and (23) to (28) and provide the transformations. The result found for $l \in [0,3]$ and $K \in [0.3]$ is postponed to Table III. It is seen that the bias exists when $l < K$ and becomes identically zero, by $l \geq K$. On the other hand, noise increases in the estimate when l increases (Table II).

Measurements of the TIE were provided for the crystal clock imbedded to the Stanford Frequency Counter SR620 using the SynPaQ III GPS Sensor. An actual TIE trend in the same clock was measured simultaneously for the Symmetricom cesium standard of frequency CsIII.

A. Simple Averaging of the Linear TIE Model

Assume that simple averaging ($l = 0$) is applied to the TIE model that is supposed to be uniform. An oscillator of the clock, however, has some amount of the frequency offset $x_2 \neq 0$ such that the model, in fact, is linear ($K = 1$). By (30) and Tables II and III, we thus have

$$\langle\varepsilon^2_{10}\rangle = \text{bias}^2_{10} + \sigma^2_{10} \tag{36a}$$

$$= \left. x_2^2\tau^2\frac{(N-1)^2}{4} + \frac{\Delta^2}{2N}\right|_{N\gg1} \tag{36b}$$

$$\cong x_2^2\tau^2\frac{N^2}{4} + \frac{\Delta^2}{2N}\,. \tag{36c}$$

Testing (36c) by (31) yields

$$N_{\text{opt}}(\tau) = \sqrt[3]{\frac{2\Delta^2}{3\tau^2 x_2^2}}\,. \tag{37}$$

TABLE III
BIAS IN THE UNBIASED FIR ESTIMATE FOR DIFFERENT TIE MODELS.

FIR filter degree l	bias_{Kl}			
	$K=0$	$K=1$	$K=2$	$K=3$
0 (uniform)	0	$-x_2\tau\frac{N-1}{2}$	$-\tau\frac{N-1}{2}\left(x_2+x_3\tau\frac{4N-5}{6}\right)$	$-\tau\frac{N-1}{2}\left[x_2+x_3\tau\frac{4N-5}{6}+x_4\tau^2\frac{(N-1)(3N-4)}{12}\right]$
1 (linear)	0	0	$-x_3\tau^2\frac{(N-1)(N-2)}{12}$	$-\tau^2\frac{(N-1)(N-2)}{12}\left(x_3+x_4\tau\frac{(3N-4)}{5}\right)$
2 (quadratic)	0	0	0	$-x_4\tau^3\frac{(N-1)(N-2)(N-3)}{120}$
3 (cubic)	0	0	0	0

If we further test (36c) by (32), we arrive at $\tau_{\text{opt}} = -\frac{\Delta^2}{3x_2^2N^3}$. Because τ cannot be negative, we take the minimum available value $\tau_{\text{opt}} = 1\text{s}$ associated with the 1PPS. An optimum N is thus

$$N_{\text{opt}}(\tau_{\text{opt}}) = \sqrt[3]{\frac{2\Delta^2}{3\tau_{\text{opt}}^2 x_2^2}}\,, \tag{38}$$

where x_2 is the fractional frequency offset in the local clock oscillator.

Fig. 2 illustrates the $N_{\text{opt}}(\tau)$ and root MSE (RMSE) found experimentally. Predicted values of $N_{\text{opt}}(\tau)$, by (37), are given here with a dash-line. As can be seen (Fig. 2b), a minimum RMSE corresponds to 1s that is also claimed by prediction. The fractional frequency offset in the clock is evaluated by averaging the first time derivative of the TIE. In our experiment, the relevant value was found to be $x_2 = 6.5\times10^{-11}$. For this measure, approximation has shown almost an ideal fit dashed in Fig. 2a. Based upon, we realize that an optimum value of N is $N_{\text{opt}}(\tau_{\text{opt}}) = 73$ that coincides with measurements.

We notice that $N_{\text{opt}}(\tau)$ specified by (37) is rather a lower bound being an approximation for $N \gg 1$. The value of N can never be lower than 2, because averaging is not available, by $N < 2$.

B. Unbiased Linear FIR Filtering of Quadratic TIE Models

In the second experiment, we apply an unbiased linear FIR filter ($l = 1$) to the assumingly linear TIE model that, by increasing N, tends to be quadratic ($K = 2$), as in Fig. 1.

Reasoning similarly and using (30), Table II, and Table III, we find the MSE

$$\langle\varepsilon_{21}^2\rangle = \text{bias}_{21}^2 + \sigma_{21}^2 \tag{39a}$$

$$= x_3^2\tau^4\frac{(N-1)^2(N-2)^2}{144} + \frac{2(2N-1)\Delta^2}{3N(N+1)}\bigg|_{N\gg1} \tag{39b}$$

$$\cong x_3^2\tau^4\frac{N^4}{144} + \frac{4\Delta^2}{3N}\,. \tag{39c}$$

Testing (39c) by (31) yields

$$N_{\text{opt}}(\tau) = \sqrt[5]{\frac{48\Delta^2}{\tau^4 x_3^2}} \tag{40}$$

and, by (32), gives $\tau_{\text{opt}} = 0$. We thus put $\tau_{\text{opt}} = 1\text{s}$. By this value, an optimum N becomes

$$N_{\text{opt}}(\tau_{\text{opt}}) = \sqrt[5]{\frac{48\Delta^2}{\tau_{\text{opt}}^4 x_3^2}}\,, \tag{41}$$

where x_3 is the linear fractional drift rate in the local clock oscillator.

Fig. 3 illustrates $N_{\text{opt}}(\tau)$ and $\text{RMSE}(\tau)$ along with the approximation of $N_{\text{opt}}(\tau)$, by (40), and $N_{\text{opt}}(\tau_{\text{opt}})$, by (41).

Again, we see (Fig. 2b) that a minimum RMSE corresponds to 1s that is also claimed by prediction. A linear fractional frequency drift rate was evaluated in the clock as $x_3 \cong 7.5\times10^{-15}/\text{s}$ by averaging the first time derivative of the fractional frequency offset x_2. By this value of x_3, (40) produces almost an ideal fit depicted in Fig. 3a by a dash-line. An optimum value of N is calculated here to be $N_{\text{opt}}(\tau_{\text{opt}}) = 1164$ that also coincides with measurements.

A comparison of Fig. 2b and Fig. 3b shows that, inherently, the RMSE is lower in the linear FIR estimate, because this filter fits the measurement better. In turn, a larger horizon is allowed for the linear filter (Fig. 3a), unlike simple averaging that is appropriate only with much shorter horizon as shown in Fig. 2a.

V. CONCLUDING REMARKS

In this paper, we found analytically an optimal horizon $N_{\text{opt}}(\tau_{\text{opt}})$ and time step τ_{opt} (in the sense of the minimum MSE) for the unbiased FIR filter applied to the TIE model of a local clock when increase in N leads to $l < K$. This typical problem occurs when one extends a horizon N in order to reduce noise in the estimate.

We showed that the minimum MSE in the estimate of the TIE corresponds to $\tau_{\text{opt}} = 1\text{s}$, meaning that thinning out the 1PPS measurements is not required for the first clock state. We notice that it is not the case for higher clock states. The lower bounds found for $N_{\text{opt}}(\tau)$ in two cases of $l = 0, K = 1$ and $l = 1, K = 2$ have demonstrated a nice fit of measurements. This means that analytical optimization of the unbiased FIR filter is available whenever the TIE model degree exceeds the filter degree, $K > l$, leading to biased estimates. Even though we have found solutions only for two particular cases, although

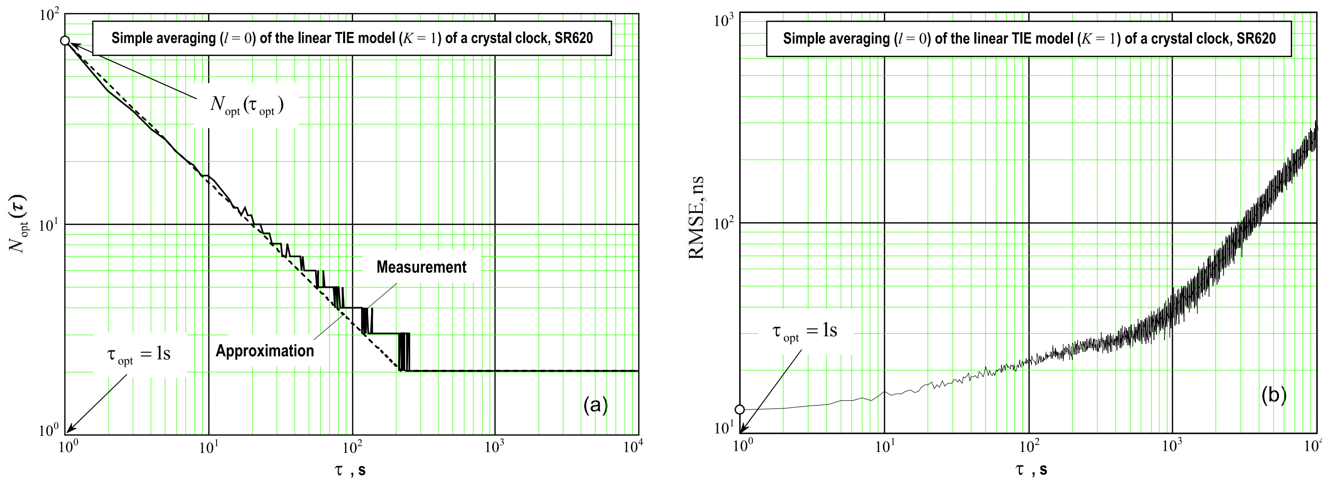

Fig. 2. Simple averaging, $l = 0$, of the linear TIE model, $K = 1$, of a local crystal clock imbedded to SR620: (a) optimal N and (b) RMSE.

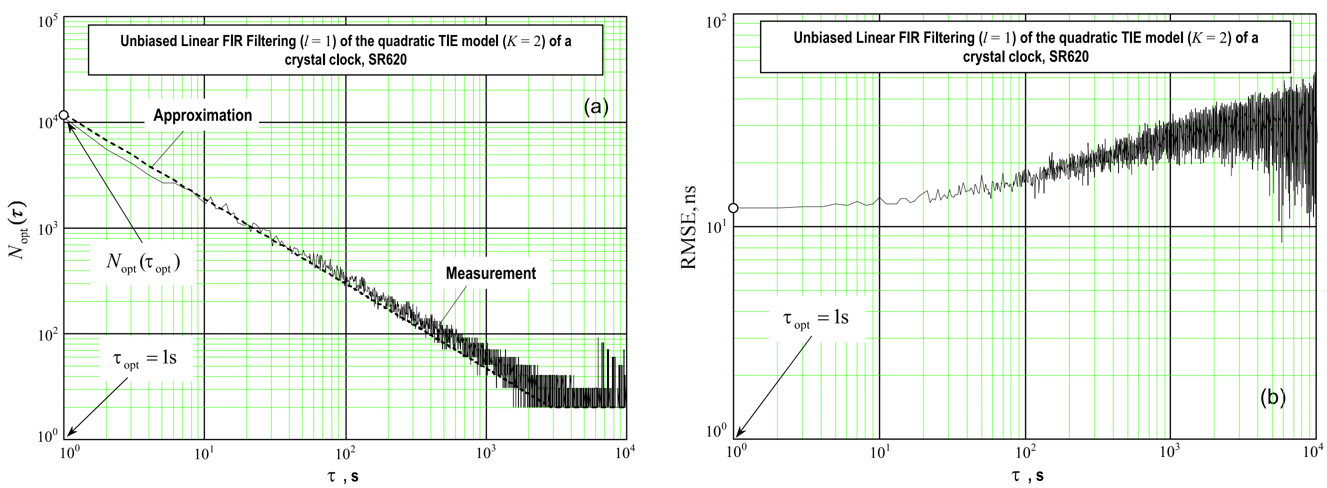

Fig. 3. Unbiased linear FIR filtering, $l = 1$, of the quadratic TIE model, $K = 2$, of a local crystal clock imbedded to SR620: (a) optimal N and (b) RMSE.

most interesting, any relation between l and K fixed by Table III can be examined that is currently under investigation.

REFERENCES

[1] W. Lewandowski, G. Petit, C. Thomas, "Precision and accuracy of GPS Time Transfer," *IEEE Trans. Instrum. Meas*. vol. 42, no. 2, pp. 474-479, 1999.

[2] F. Meyer, "Common-view and melting-pot GPS time transfer with the UT+," *Proc. 32nd Annu. PTTI Mtg*., Reston, VA, USA, 2000, pp.147-156.

[3] Yu. S. Shmaliy, "An Optimal FIR Filter for Linear TIE Models of Local Clocks", in *these Proceedings*.

[4] R. M. Hambly, T. A. Clark, "Critical evaluation of the Motorola M12+ GPS timing receiver vs. the master clock at the United States Naval Observatory," Washington, DC, *Proc. 34th Annu. PTTI Mtg*., Reston, VA, USA, 2002, pp. 109-115.

[5] T. A. Clark and R. M. Hambly, "Improving the performance of low cost GPS Timing Receivers" in *Proc. 38th Annu. Precise Time and Time Interval Meeting,* 2006 (to be published).

[6] Yu. S. Shmaliy, "A simple optimally unbiased MA filter for timekeeping," *IEEE Trans. on Ultrason., Ferroel. and Freq. Contr*. vol. 49, no. 6, pp. 789-797, 2002.

[7] Yu. S. Shmaliy, "An unbiased FIR filter for TIE model of a local clock in applications to GPS-based timekeeping," *IEEE Trans. Ultrason., Ferroel., and Freq. Contr.*, vol. 53, no. 5, pp. 862-870, 2006.

[8] J. Further, A. Moudrak, A. Konovaltsev, J. Hammesfahr, and H. Denks, "Time dissimination and common view time transfer with GALILEO: How accurate will it be?" in *Proc. 35th Annu. Precise Time and Time Interval Meeting,* 2003, pp. 185-198.

[9] J. Munoz-Diaz, Yu. S. Shmaliy, L. Arceo-Miquel, and O. Ibarra-Manzano, "Investigation of an optimum sampling interval for a local clock TIE model with an Unbiased FIR filtering algorithm," in *Proc. IEEE Int. Freq. Control Symp.*, 2006, pp. 599-603.

[10] Yu. S. Shmaliy, O. Ibarra-Manzano, L. Arceo-Miquel, and J. Munoz-Diaz, "A thinning algorithm for GPS-based unbiased FIR estimation of a clock TIE model," *Measurement* (to be published).